[redacted]

Chapter 5

Still Talking about Linear 2D Systems.

we know $\frac{dx}{dt} = kx$; $x(0) = x_0$

has the solution $x(t) = x_0 e^{kx}$

we know for real k →

$k > 0$	$x \to \infty$	; $t \to \infty$
$k < 0$	$x \to 0$	; $t \to \infty$
$k = 0$	$x = x_0$	for all time

we can generalize this for systems

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = A \qquad \dot{\mathbf{x}} = A\mathbf{x}$$
$$\mathbf{x}(0) = \mathbf{x}_0$$

~~has solution~~ has a general solution

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2$$

λ_1, λ_2 ⇒ eigenvalues

$\mathbf{v}_1$, $\mathbf{v}_2$ ⇒ eigenvectors (straight line traj)

for any initial condition

~~$\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$~~

find the eigenvalues by solving the characteristic equation $\det(A - \lambda I) = 0$

$$\det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = 0$$

$$(a-\lambda)(d-\lambda) - cb = 0$$

$$\lambda^2 - \tau\lambda + \Delta = 0$$

$$\tau = \text{trace}(A) = \Sigma \text{ diagnols} = a + d$$
$$\Delta = \det(A) = ad - bc$$

$$\lambda_1 = \frac{\tau + \sqrt{\tau^2 - 4\Delta}}{2} \qquad \lambda_2 = \frac{\tau - \sqrt{\tau^2 - 4\Delta}}{2}$$

find the corresponding eigenvectors that satisfies

$$A\mathbf{v} = \lambda\mathbf{v}$$
$$(A - I\lambda)\mathbf{v} = 0$$

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$$

$$\begin{pmatrix} a-\lambda_1 & b \\ c & d-\lambda_1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \rightarrow \mathbf{v}_1$$

$$\begin{pmatrix} a-\lambda_2 & b \\ c & d-\lambda_2 \end{pmatrix}\begin{pmatrix} v_3 \\ v_4 \end{pmatrix} = 0 \rightarrow \mathbf{v}_2$$

3

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 e^{\lambda_1 t} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + c_2 e^{\lambda_2 t} \begin{pmatrix} v_3 \\ v_4 \end{pmatrix}$$

$$x_1 = c_1 v_1 e^{\lambda_1 t} + c_2 v_3 e^{\lambda_2 t}$$

$$x_2 = c_1 v_2 e^{\lambda_1 t} + c_2 v_4 e^{\lambda_2 t}$$

(c_1, c_2) must satisfy $x_1(0) = x_{10}$
$x_2(0) = x_{20}$

$$x_{1,0} = c_1 v_1 (1) + c_2 v_3 (1)$$
$$x_{2,0} = c_1 v_2 (1) + c_2 v_4 (1)$$

eigenvalues tell you how you move; (e.g. decay, grow?) + how quickly

eigenvectors tell you the direction.

e.g. in book...

$$\frac{dx}{dt} = x + y$$

$$\frac{dy}{dt} = 4x - 2y$$

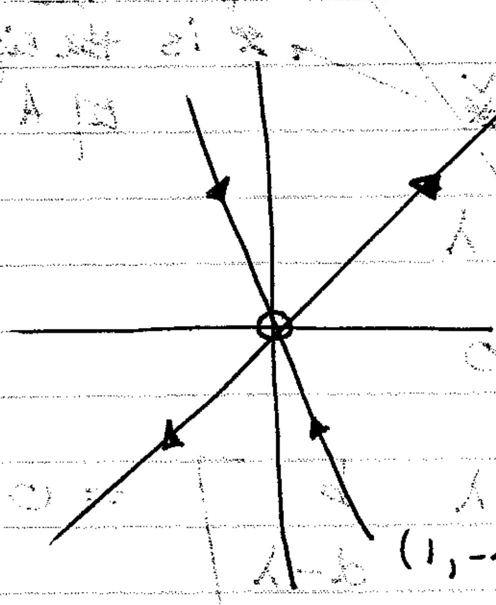
$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

these 2 are equivalent.

has the solution

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t} \quad (4)$$

how does this look on our phase portrait?



$(1,1) = v_1$ is our unstable manifold. $\lambda_1 > 0 \;\therefore$ blows up as $t \to \infty$

$\boxed{\text{figure } 5.2.2}$

$(1,-4) = v_2$ is our stable manifold

$\lambda_2 < 0 \;\therefore$ decays $\to 0$ as $t \to \infty$

- the origin $(0,0)$ is a fixed point.
  a saddle node

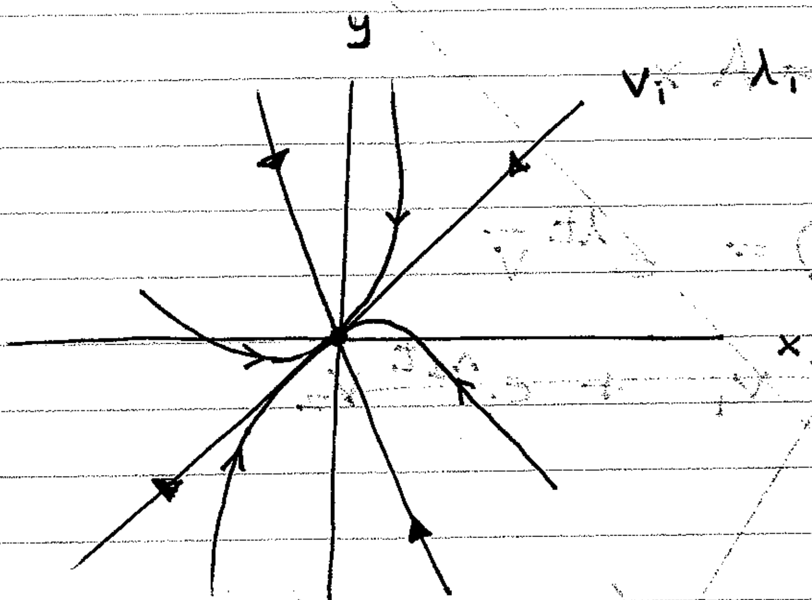
→ node you only get to the origin if you travel along the stable manifold.

→ just like our last class example; 5.1.2 in book. but in that case our system was uncoupled, so the eigenvectors aligned w/ the x & y axes.

$\dot{x} = ax$

$\dot{y} = -y$

if both λ were $-$; then the origin would be a stable node.



v_1 $\lambda_1 = -2$; slow direction

v_2 $\lambda_2 = -3$; fast direction

$|\lambda_1| < |\lambda_2|$ so travel along v_2 is faster than ~~th~~ travel along v_1

$$\lambda_1 = \frac{\tau + \sqrt{\tau^2 - 4\Delta}}{2} \qquad \lambda_2 = \frac{\tau - \sqrt{\tau^2 - 4\Delta}}{2}$$

what if λ_1 is complex?

$$\alpha = \tau/2$$

$$\omega = \frac{1}{2}\sqrt{4\Delta - \tau^2}$$

$$e^{(\alpha \pm i\omega)t}$$

$$e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$$

So $e^{\alpha t}\left(\cos(\omega t) + i\sin(\omega t)\right)$

so get <u>oscillations</u>!